Midterm - Optimization (2019-20)

Attempt all questions. There are a total of 27 points, the maximum you can score is 25.

Time: 2 hours.

1. Solve the system $\mathbf{Ax} = \mathbf{b}$ (first check whether it is consistent, and then find the general solution if the system is consistent). Also find for $\mathbf{A}$, a g-inverse, the rank, a rank factorisation and a basis of the null space. [**4 points**]

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 0 & 4 \\ 3 & 1 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$$

2. Consider the problem: minimize $2x_1 + 3|x_2 - 10|$ subject to $|x_1 + 2| + |x_2| \leq 5$. Reformulate this as a linear programming problem. [**3 points**]

3. For each of the statements below, state whether it is true or false. If true, prove it and if false, give a counterexample.

   (a) Every polyhedron has an extreme point. [**3 points**]

   (b) Consider the problem of minimizing $\mathbf{c}^T\mathbf{x}$ over a polyhedron P. If there is more than one optimal solution, then there are uncountably many optimal solutions. [**3 points**]

   (c) Consider the problem of minimizing $\mathbf{c}^T\mathbf{x}$ over a polyhedron P. The set of optimal solutions is bounded. [**3 points**]

   (d) Every basic solution of a polyhedron is feasible. [**3 points**]

4. Suppose that the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_i^T\mathbf{x} \geq b_i,\ i = 1, 2, \cdots, m\}$ is nonempty. Show that if the polyhedron P does not contain a line, then it has at least one extreme point. [**4 points**]

5. Consider the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_i^T\mathbf{x} \geq b_i,\ i = 1, \cdots, m\}$. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are distinct basic feasible solutions that satisfy $\mathbf{a}_i^T\mathbf{u} = \mathbf{a}_i^T\mathbf{v} = b_i,\ i = 1, \cdots, n-1$, and that the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_{n-1}$ are linearly independent. Let $L = \{\lambda\mathbf{u} + (1-\lambda)\mathbf{v} : 0 \leq \lambda \leq 1\}$ be the segment that joins $\mathbf{u}$ and $\mathbf{v}$. Prove that $L = \{\mathbf{z} \in P : \mathbf{a}_i^T\mathbf{z} = b_i,\ i = 1, \cdots, n-1\}$. [**4 points**]